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SKEW-QUADRUPOLE FIELD AND HORIZONTAL-VERTICAL COUPLING IN THE MAIN RING

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I. SOURCES OF SKEW-QUADRUPOLE FIELD

A. Inner-coil End of Bending Magnet

The inner coil (4 turns) goes around the same side of the gap at the end of a bending magnet, thereby producing a skew-quadrupole field and a longitudinal field. A crude estimate of the skew-quadrupole field is as follows:

The inner-coil end can be replaced by a normal symmetric end (2 turns each going around the top and the bottom sides of the gap) which produces no skew-quadrupole and longitudinal fields plus a 2-turn coil looping the gap. For an estimate of $B_x(x)$ on the midplane, we can consider the gap in the yoke closed (filled with iron) and replace the loop current by only the 2 vertical sides, which are approximated by line currents. Looking down from above the equivalent end looks like

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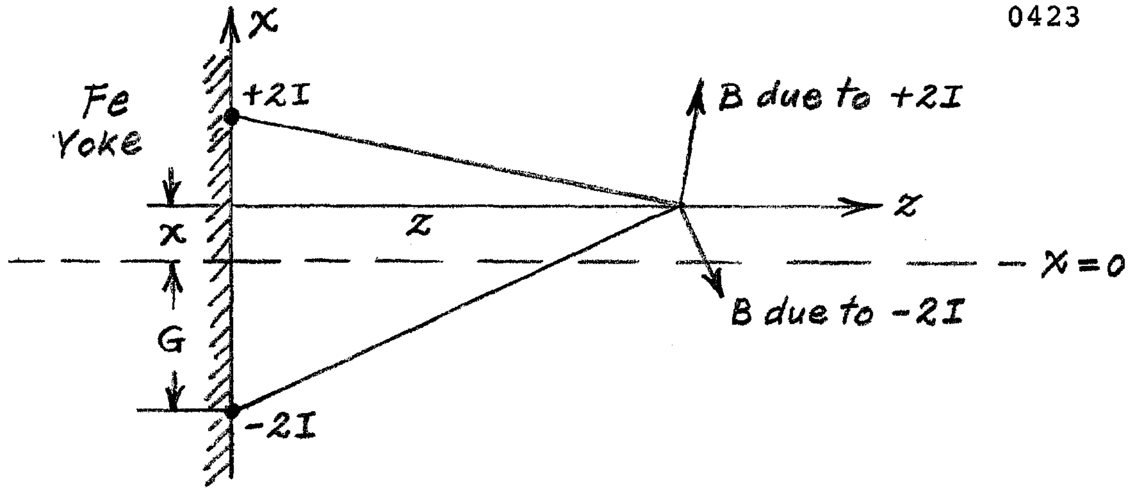

The horizontal transverse field B_x is, now, given by

$$B_x = \frac{z}{\sqrt{(G-x)^2+z^2}} \cdot \frac{4(2I)}{\sqrt{(G-x)^2+z^2}} - \frac{z}{\sqrt{(G+x)^2+z^2}} \cdot \frac{4(2I)}{\sqrt{(G+z)^2+z^2}}$$

or

$$\int_0^\infty B_x dz = 4\,I \left[\ell n \frac{(G-x)^2+z^2}{(G+x)^2+z^2} \right]_0^\infty = -4\,I\,\ell n \frac{\left(1-\frac{x}{G}\right)^2}{\left(1+\frac{x}{G}\right)^2}$$

$$\cong 16\,I\,\frac{x}{G} \qquad \text{for } \frac{x}{G} << 1.$$

Thus

$$\int_{\text{across end}} \frac{\partial B_x}{\partial x}\, dz = 16 \frac{I}{G} = \frac{B_o g}{4\pi G}$$

for a B2 magnet with a 16-turn coil, where $B_o = B_y$ inside magnet, g = vertical gap = 2" (for B2), and G ≅ 3" (for B2). This gives

$$k \equiv \frac{1}{B_o} \int \frac{\partial B_x}{\partial x}\, dz = \frac{1}{4\pi}\,\frac{2}{3} = 0.053.$$

Closing the yoke gap tends to make this an overestimate. The measured value is

$$k \equiv \frac{1}{B_o} \int \frac{\partial B_x}{\partial x}\, dz = 0.041.$$

The + and - ends are mostly paired off in the design. Normally we have only one unpaired bending magnet (2 ends) per sector. The 12 unpaired ends, then, give

$$K = 12k = 0.49.$$

This skew-quadrupole field tracks the bending field all the way up.

B. Roll Error of Quadrupole Magnet

With a roll angle of ε we have

$$\frac{\partial B_x}{\partial x} = 2\varepsilon \frac{\partial B_y}{\partial x} .$$

For a 7'-quad (length = ℓ = 2.1336 m) and ε = 1 mrad we have

$$k \equiv 2\varepsilon\, \ell\, \frac{1}{B_o} \frac{\partial B_y}{\partial x} = 0.060.$$

The roll error is mainly due to the difference between the field axis and the geometrical axis and amounts to an rms angle of about 10 mrad. For ~200 quadrupoles we get

$$K = 10\sqrt{200}\ k = 8.5.$$

This skew-quadrupole field also tracks the bending field and is, by far, the largest contribution.

C. Off-Center Sextupole Field in Bending Magnet

Since the beam is not centered in the bending magnet the sextupole component of the remanent field produces a quadrupole and a skew-quadrupole field at the beam. The scalar magnetic potential of the remanent sextupole field can be written as

$$\Phi_0 = \frac{B''}{6}\,(3x^2y - y^3)$$

when $B'' \equiv \partial^2 B_y/\partial x^2$ (on center). If the beam is centered at (x_0, y_0) the field at the beam is given by

$$\Phi = \frac{B''}{6}\left[3(x-x_0)^2(y-y_0) - (y-y_0)^3\right]$$

$$= \Phi_0 - B''x_0xy - \frac{B''}{2}\,y_0(x^2-y^2)$$

$$+ \text{ higher-degree terms in } x_0 \text{ and } y_0.$$

Thus, a horizontal displacement x_0 produces a normal quadrupole term $- B''x_0xy$ and a vertical displacement y_0 produces a skew-quadrupole term

$$\Phi_{skew} = -\frac{B''}{2}\,y_0(x^2-y^2)$$

which gives

$$\frac{\partial B_x}{\partial x} = -B'' y_o .$$

At injection $B_o = 0.36$ kG, $B'' = -0.4$ kG/m^2 with $y_o = 1$ mm $= 10^{-3}$ m, we get for each bending magnet (length = ℓ = 6.0706 m)

$$k = \frac{\ell}{B_o} \frac{\partial B_x}{\partial x} = -\frac{B''}{B_o} y_o \ell = 0.0067.$$

For an rms y_o of 10 mm (including both bending-magnet misalignment and closed-orbit distortion) and 774 bending magnets we have

$$K = 10\sqrt{774}\, k = 1.9 \qquad \text{(at injection)}.$$

Here we have assumed that y_o is totally uncorrelated from magnet to magnet. In actuality, there is some correlation among y_o values, especially among those in the 4 bending magnets in a half-cell. This increases K. On the other hand, there is also some correlation between y_o values in the 4 bending magnets and that in the trim sextupole magnet in the same half-cell. This reduces K. As a crude estimate the above value is probably not too bad.

This skew-quadrupole field, of course, does not track the bending field. Hence K decreases as $\frac{1}{B_o}$.

II. HORIZONTAL-VERTICAL COUPLING

The skew-quadrupole field excites the coupling resonance $\nu_x - \nu_y = 0$. The approximate coupled equations are

$$\begin{cases} x'' + \nu_x^2 x = Cy \\ y'' + \nu_y^2 y = Cx \end{cases} \qquad \text{prime} \equiv \frac{d}{d\theta}$$

where

$$C = \frac{R^2}{B\rho} \left\langle \frac{\partial B_x}{\partial x} \right\rangle = \frac{R^2}{\rho} \frac{K}{2\pi R} = 0.213\ K$$

for the main ring. These coupled equations can be solved by transforming to the normal coordinates u and v given by

$$\begin{cases} u = x - ay \\ v = ax + y \end{cases} \quad \text{or} \quad \begin{cases} x = \dfrac{u + av}{1 + a^2} \\ y = \dfrac{v - au}{1 + a^2} \end{cases}$$

where

$$a \equiv \sqrt{1 + \xi^2} - \xi \qquad \xi \equiv \frac{\nu_x^2 - \nu_y^2}{2C}$$

The normal coordinates u and v are uncoupled and obey the equations

$$\begin{cases} u'' + \nu_u^2 u = 0 \\ v'' + \nu_v^2 u = 0 \end{cases} \quad \text{where} \quad \begin{cases} \nu_u^2 = \nu_x^2 + aC \\ \nu_v^2 = \nu_y^2 - aC \end{cases}$$

There is no need to write down the general solutions for x and y. All important information can be obtained from the special solution for $x = 1$ $x' = 0$ $y = y' = 0$ at $\theta = 0$. This is

$$\begin{cases} x = \dfrac{1}{1+a^2}\,(\cos\,\nu_u\theta + a^2\,\cos\,\nu_v\theta) \\[2ex] y = \dfrac{a}{1+a^2}\,(\cos\,\nu_v\theta - \cos\,\nu_u\theta) \end{cases}$$

Let

$$\begin{cases} \nu_u \equiv \bar{\nu} + \frac{\mu}{2} \\[2ex] \nu_v \equiv \bar{\nu} - \frac{\mu}{2} \end{cases} \qquad \begin{cases} \bar{\nu} \equiv \dfrac{\nu_u + \nu_v}{2} \\[2ex] \mu \equiv \nu_u - \nu_v \end{cases}$$

and we get

$$\begin{cases} x = \left(\cos\,\frac{\mu}{2}\theta\right)\,\cos\,\bar{\nu}\theta - \left(\dfrac{1-a^2}{1+a^2}\,\sin\,\frac{\mu}{2}\theta\right)\,\sin\,\bar{\nu}\theta \\[2ex] y = \left(\dfrac{2a}{1+a^2}\,\sin\,\frac{\mu}{2}\theta\right)\,\sin\,\bar{\nu}\theta \end{cases}$$

Thus, we see that x and y are amplitude-modulated oscillations. The amplitudes A_x and A_y are given by

$$\begin{cases} A_x^2 = \dfrac{1}{\left(1+a^2\right)^2}\left(1 + a^4 + 2a^2\,\cos\,\mu\theta\right) \\[2ex] A_y^2 = \dfrac{2a^2}{\left(1+a^2\right)^2}\left(1 - \cos\,\mu\theta\right) \end{cases} \qquad A_x^2 + A_y^2 = 1$$

At $\theta = 0$, $A_x = 1$ $A_y = 0$ and at

$$\mu\theta = \pi \qquad \text{or} \qquad \theta = \frac{\pi}{\mu} = \frac{\pi}{\nu_u - \nu_v}$$

$$\begin{cases} A_x = A_{x_{min}} = \dfrac{1-a^2}{1+a^2} = \dfrac{\xi}{\sqrt{1+\xi^2}} \\[2ex] A_y = A_{y_{max}} = \dfrac{2a}{1+a^2} = \dfrac{1}{\sqrt{1+\xi^2}} \end{cases}$$

For an example we take

$$C = 2 \qquad \text{or} \qquad K = \frac{2}{0.213} = 9.39$$

and

$$\nu_x = 20.3 \qquad \nu_y = 20.2$$

which are not unreasonable values. We get

$$\xi = 1.0125 \qquad a = 0.41$$

$$\begin{cases} \nu_u = 20.32 \\ \nu_v = 20.18 \end{cases} \quad \text{or} \quad \begin{cases} \bar{\nu} = 20.25 \\ \mu = 0.14 \end{cases}$$

$$\begin{cases} A_{x\ min} = 0.71 \\ A_{y\ max} = 0.70 \end{cases} \quad \text{occurring at } \theta = 3.56(2\pi)$$

(3.56 revolutions)

This is very strong coupling indeed.

To compensate for this, we need the equivalent of 0.67 m length of the main ring quadrupole $\left(\frac{1}{B_o} \frac{\partial B_y}{\partial x} = 14 \text{ m}^{-1}\right)$. This total skew-quadrupole strength should be divided into many smaller units appropriately located around the ring to eliminate not only the zeroth harmonic but also the 40th and the 41st harmonics (to avoid exciting the $\nu_x + \nu_y = 40$ and $\nu_x + \nu_y = 41$ resonances). Properly, one should eliminate all 4 elements of the 2x2 x-y coupling transfer matrix over a revolution. In view of the rather fast rate of couple-over (3.56 revolutions) it would be more preferable to eliminate the coupling transfer matrix for each sector, but this will require a large amount of detailed precision adjustment.